Name: RED ID:

Fall 2024 Math 245 Final Exam

Please read the following directions:

Please write legibly, with plenty of white space. Please **print** your name and REDID in the designated spaces above. Please fit your answers into the designated areas; material outside the designated areas (such as on this cover page) will not be graded. To get credit, you must also show adequate work to justify your answers. If unsure, show the work. The first four questions are worth 7-13 points, and the remaining sixteen questions are worth 9-18 points. The maximum possible score is $4 \times 13 + 16 \times 18$, for a total of 340 points. The use of notes, books, calculators, or other materials on this exam is strictly prohibited, except you may bring one 8.5"x11" page (both sides) with your handwritten notes. If you need scratch paper, you may use any blank space on your note sheet and on this front page. This exam will begin at 10:30 and will end at 12:30; pace yourself accordingly. Good luck!

Special exam instructions for HH-130:
1. Please don't leave empty seats in the middle of the classroom – they will only get filled after the exam starts, which is better for nobody.
2. Please stow all bags/backpacks/boards at the front of the room. All contraband, except phones, must be stowed in your bag. All smartwatches and phones must be silent, non-vibrating, and either in your pocket or stowed in your bag.
3. Please remain quiet to ensure a good test environment for others.
4. Please keep your exam on your desk; do not lift it up for a better look.
5. If you have a question or need to use the restroom, please come to the front. Bring your exam. I cannot come to you unless you are sitting by an aisle, sorry.
6. If you are done and want to submit your exam and leave, please wait until one of the three designated exit times, listed below. Please do **NOT** leave at any other time. If you are sure you are done, just sit and wait until the next exit time, with this cover sheet visible.

Designated exam exit times:
10:50 "See you next semester"
11:10 "I wish I had studied more"
11:30 "One extra hour of drinking – worth it"
11:50 "Maybe this will be good enough"
12:10 "There is nothing more in my brain, let me out of here"
12:30 "I need every second I can get"

Problem 1. Carefully state the following definitions:

a. prime

b. floor

Problem 2. Carefully state the following theorems:

a. Conditional Interpretation Theorem

b. Trivial Proof Theorem

Problem 3. Carefully state the following definitions:

a. power set

b. equivalence relation

Problem 4. Carefully state the following definitions:

a. interval poset

b. range

Problem 5. Carefully state and prove the Double Negation Theorem.

Problem 6. Let p, q, r be propositions. Without truth tables, prove $q \to p, r \to q \vdash r \to p$.

Problem 7. Prove or disprove: $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \exists z \in \mathbb{Z},\ (x < y) \to (x < z < y)$.

Problem 8. Prove or disprove: $\forall x \in \mathbb{Z}, !y \in \mathbb{Z},\ |x+1| = |y+2|$.

Problem 9. Let F_n denote the Fibonacci numbers. Prove $\forall n \in \mathbb{N}_0,\ F_n < \left(\frac{19}{10}\right)^n$.
HINT: $1.9^2 = 3.61$

Problem 10. Let a_n, b_n, c_n be sequences with $a_n = O(b_n)$ and $b_n = O(c_n)$. Prove $a_n = O(c_n)$.
NOTE: This happens to be a theorem in the book. Do not use it to prove itself!

Problem 11. Prove or disprove: For all sets R, S, T, if $R \cup S \subseteq S \cap T$, then $R \subseteq T$.

Problem 12. Find sets A, B, C with $|A| > |B| > |C|$ and $(A \times B) \times C = A \times (B \times C)$.
Give the sets explicitly, in list notation, and justify each of the properties.

Problem 13. Let R be a relation on set S. Suppose $\forall x, y \in S,\ xRy \to yRx$.
Prove $\forall x, y \in S,\ xRy \leftrightarrow yRx$.
This connects the two definitions of symmetry. Do not use this connection to prove itself!

Problem 14. Let R be a relation on set S. Suppose $R \circ R \subseteq R$. Prove that R is transitive.

Problem 15. Compute which $x \in [0, 5)$ satisfies $2^{65} \equiv x \pmod 5$. Fully justify each step.

Problem 16. Let R be an equivalence relation on set S. Let $x, y \in S$. Suppose that $[x] \subseteq [y]$.
Prove $[y] \subseteq [x]$.

Problem 17. Let R be a partial order on set S. Let $T \subseteq S$. Suppose a, b are both least in T. Prove that $a = b$.

Problem 18. Consider the divisibility relation $|$ on $S = \{2, 3, 4, 6\}$. Find all linear extensions of $|$ on S.

Problem 19. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sin x$. Determine, with proof, whether or not f is injective. Also, determine, with proof, whether or not f is surjective.

Problem 20. Prove or disprove: For any functions f, g on $\mathbb{R}$, if both f and $g \circ f$ are injective, then g must be injective.